Literature Review: 0DTE Options Pricing

Newt Nguyen
The Hong Kong University of Science and Technology

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1 Vocabulary

There are some new vocabulary to know:

- Affine model: a model where the drift, variance, jump intensities are linear functions of the state variables. For example, the drift term $u(X_t) = a + bX_t$ is linear in X_t.
- Skewness (3rd moment): measure asymmetry in returns. Kurtosis (4th moment): measure tail fatness and peakiness
- Quadratic Covariation Process: denotes $[X, Y]_t$, measures the joint variability of two stochastic processes X_t and Y_t over time.
  - Ito's Lemma for 2 Ito processes X_t and Y_t states that:

$$d(X_tY_t) = X_t dY_t + Y_t dX_t + d[X, Y]_t$$

  - Definition of quadratic covariation: $[X, Y]_t = \lim_{\|\Pi\| \to 0} \sum_{i=1}^{n} (X_{t_i} - X_{t_{i-1}})(Y_{t_i} - Y_{t_{i-1}})$

2 Main Objective

In this paper, our main objective is:

- Present novel closed-form pricing formula that accurately capture the 0DTE implied-volatility surface. We will use a local in-time approach, relying on Edgeworth-like expansions of the log-return characteristic function that is suited to price ultra-short-tenor instruments.
- The expansions provide skewness and kurtosis adjustments which depend on the underlying non-affine return characteristics in closed form.

3 Non-technical part

A few non-technical parts to know in this paper:

- 0DTE offer opportunities to traders who can capitalize on very short-term directional bets (e.g. macroeconomic announcements). Since we want to price short-tenor options and generalize it, we only impose mild assumptions:
  - The price process features both continuous and discontinuous components X_t^c is continuous part and X_t^d is discontinuous part.
  - All characteristics driving the dynamics are unrestricted processes only required to satisfy technical and smoothness conditions.
  - We need to evaluate Q-characteristic function of the logarithmic price process over a short horizon, to find out the distribution of the price process over that horizon.
- We will provide 2 closed-form expressions for the pricing of characteristic function over small-τ intervals.

4 First closed-form expression

In this specification, the discontinuity part is modeled as a known characteristic function, and the continuity part is modeled around the Gaussian characteristic function. Basically, we will be having

$$C^{\log x}(u,\tau) = E_t^Q[e^{iu(\log X_{t+r} - \log X_t)}] = C^{\log X^c}(u,\tau)C^{\log X^d}(u,\tau)$$

The form of discontinuity part will be pre-determined. However, the form of continuity part will be determined by the Edgeworth expansions approach.

4.1 Local in-time Edgeworth expansions approach

Definition 1. An adapted, real stochastic W-Ito process X_t is defined as being W-differentiable (differentiable with respect to the standard Brownian motion W) if it admits the representation:

$$dX_t = a_t^X dt + b_t^X dW_t$$

In here, the scalar processes a_t^X and b_t^X are all adapted processes.

Some explanations (understand by myself), is that if $a_t^X = a$, $b_t^X = b$, then the process is W-differentiable of degree 1 (meaning that it only differentiates 1 time with respect to Brownian motion).

It also means that the differentiability of the process X_t seems to depend on the differentiability of the process a_t^X and b_t^X. It also seems that if the process a_t^X and b_t^X is (W)-differentiable, then X_t will be (W + 1)-differentiable. Thus, we have definition 2.

Definition 2. An adapted, real stochastic W-Ito process X_t is k-times W-differentiable if it admits the representation:

$$dX_t = a_t^X dt + b_t^X dW_t$$

$$dS_t^j = A_t^{X(j)} dt + B_t^{X(j)} dW_t$$

where the 2^j-dimensional vector process S_t^j is defined as $S_t^j = (A_t^{X(j-1)}, B_t^{X(j-1)})^T$, and we have the base case $(A_t^{X(0)}, B_t^{X(0)})^T = (a_t^X, b_t^X)^T$.

Say, D_W^k the family of k times W-differentiable processes.

Some examples from the paper:

Example 1. Consider a real function f which is differentiable $2k$ times in its domain. Then the process $f(W_t)$ is in $D_W^{(k+1)}$.

We can prove this by Ito's lemma:

$$df(W_t) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt$$

This implies $f(W_t)$ is in $D_W^{(2)}$. Keep iterating, we will have $f(W_t) \in D_W^{(2)}$.

Since we are interested in the characteristic function expansions, we will need to evaluate:

$$f_{X_t}(u) = \mathbb{E}_0[e^{iuW_t}\chi_t]$$

Nice that I know how to type this new notations. We call the above W-transform. Thus, we have the first lemma:

Lemma 1. Consider a W-differentiable process χ_t defined above. Its W-transform satisfies:

$$\mathbb{E}_0[e^{iuW_t}\chi_t] = e^{\frac{-u^2}{2}t} + (\chi_0 + \int_0^t e^{\frac{u^2}{2}s}\mathbb{E}_0[e^{iuW_s}(d\chi_s + iud[\chi, W]_s)$$

where $[\chi, W]_t$ is the quadratic covariation process between χ_t and W_t.

The proof, according to the paper

Using Ito's Lemma:

$$df = f'(B_t)dB_t + \frac{1}{2}f^{(2)}(B_t)dt$$

We would then have:

$$d(e^{iuW_s}) = iue^{iuW_s}dW_s - \frac{1}{2}u^2 e^{iuW_s}ds$$

By the Itô product rule for two semimartingales X_t and Y_t, we have:

$$d(X_tY_t) = X_t\, dY_t + Y_t\, dX_t + d[X,Y]_t.$$

Integrating both sides from 0 to t yields:

$$\int_0^t d(X_sY_s) = \int_0^t X_s\,dY_s + \int_0^t Y_s\,dX_s + \int_0^t d[X,Y]_s.$$

The left-hand side evaluates as:

$$\int_0^t d(X_sY_s) = X_tY_t - X_0Y_0$$

$$X_tY_t = X_0Y_0 + \int_0^t X_s\,dY_s + \int_0^t Y_s\,dX_s + \int_0^t d[X,Y]_s$$

For $X_t = \chi_t$ and $Y_t = e^{iuW_t}$:

$$e^{iuW_t}\chi_t = \chi_0 e^{iuW_0} + \int_0^t e^{iuW_s}\,d\chi_s + \int_0^t \chi_s\,d(e^{iuW_s}) + \int_0^t d[\chi, e^{iuW}]_s \tag{1}$$

$$= \chi_0 + \int_0^t e^{iuW_s}\,d\chi_s + \int_0^t \chi_s\,d(e^{iuW_s}) + \int_0^t d[\chi, e^{iuW}]_s \tag{2}$$

since $W_0 = 0$ and $e^0 = 1$.

Notice that we only have an Ito's lemma derivation above. Thus, I can write:

$$e^{iuW_t}\chi_t = \chi_0 + \int_0^t e^{iuW_s}d\chi_s + \int_0^t \chi_s(iue^{iuW_s}dW_s - \frac{1}{2}u^2e^{iuW_s}ds) + \int_0^t d[\chi, e^{iuW}]_s$$

We put our focus on the quadratic covariation process. We can further simply it to be:

$$d[\chi, e^{iuW}]_t = iue^{iuW_t}d[\chi, W]_t$$

Let $f_{\chi_t}(u) := \mathbb{E}_0[e^{iuW_t}\chi_t]$. Basically, we are taking expectation from the two sides of the equation. Thus, we will then have:

$$f_{\chi_t}(u) = \chi_0 + \mathbb{E}_0\left[\int_0^t e^{iuW_s}d\chi_s\right] - \frac{u^2}{2}\int_0^t f_{\chi_s}(u)ds + iu\mathbb{E}_0\left[\int_0^t e^{iuW_s}d[\chi, W]_s\right]$$

Remember that: **Martingale Property**: For any adapted process H_s satisfying $\mathbb{E}\left[\int_0^t H_s^2 ds\right] < \infty$,

$$\mathbb{E}\left[\int_0^t H_s dW_s\right] = 0.$$

Here, $H_s = iu\chi_s e^{iuW_s}$

Some searches told me that this is the Volterra-type integral equation. We can denote:

$$f(t) = g(t) - \frac{u^2}{2}\int_0^t f(s)ds, \quad g(t) := \chi_0 + \int_0^t \mathbb{E}_0\left[e^{iuW_s}(d\chi_s + iud[\chi, W]_s)\right]$$

Differentiating both sides:

$$\frac{d}{dt}f_{\chi_t}(u) = \mathbb{E}_0\left[e^{iuW_t}(d\chi_t + iud[\chi, W]_t)\right] - \frac{u^2}{2}f_{\chi_t}(u).$$

Rearranged as:

$$f'(t) + \frac{u^2}{2} f(t) = g'(t), \quad \text{where } g(t) = \chi_0 + \int_0^t \mathbb{E}_0 \left[e^{iuW_s} (d\chi_s + iud[\chi, W]_s) \right].$$

In order to solve it, we multiply through by the integrating factor $e^{\frac{u^2}{2}t}$:

$$e^{\frac{u^2}{2}t} f'(t) + \frac{u^2}{2} e^{\frac{u^2}{2}t} f(t) = e^{\frac{u^2}{2}t} g'(t).$$

$$\frac{d}{dt}\left(e^{\frac{u^2}{2}t} f(t) \right) = e^{\frac{u^2}{2}t} g'(t).$$

$$e^{\frac{u^2}{2}t} f(t) - f(0) = \int_0^t e^{\frac{u^2}{2}s} g'(s) ds.$$

Since $f(0) = \chi_0$ and by the Fundamental Theorem of Calculus:

$$e^{\frac{u^2}{2}t} f(t) = \chi_0 + \int_0^t e^{\frac{u^2}{2}} s dg(s).$$

Substitute back for $g(t)$ and divide by the integrating factor:

$$f(t) = e^{-\frac{u^2}{2}t} \left(\chi_0 + \int_0^t e^{\frac{u^2}{2}s} \mathbb{E}_0 \left[e^{iuW_s} (d\chi_s + iud[\chi, W]_s) \right] \right).$$

We have a proposition that we will take it for granted:

Proposition 1. Define the multiple integral: (probably not important? April 4th, 2025)

$$I_k(u,t) = \int_0^t \int_0^{s_1} \dots \int_0^{s_{k-1}} e^{\frac{u^2}{2}s_k} ds_1 \dots ds_k$$

Then, it will be equal to:

$$I_k(u,t) = \frac{2^k}{u^{2k}} \left(e^{\frac{u^2}{2}t} - \sum_{j=0}^{k-1} \frac{1}{j!} \left(\frac{u^2}{2} t \right)^j \right) = \frac{2^k}{u^{2k}} e^{\frac{u^2}{2}t} \left(1 - \frac{\Gamma\left(k, \frac{u^2}{2}t\right)}{(k-1)!} \right),$$

From Lemma 1, we are expecting to have a corollary. The corollary is stated as below:

Corollary 1. Given Lemma 1, if $\chi_t \in D_w^{(1)}$, then we have:

$$\mathbb{E}_0 \left[e^{iuW_t} \chi_t \right] = e^{-\frac{u^2}{2}t} \left(\chi_0 + \int_0^t \mathbb{E}_0 \left[e^{iuW_s} (a^X + iub^X) \right] e^{\frac{u^2}{2}s} ds \right)$$

Reason is simple, since χ_t is only 1-differentiable, then we would have the following:

$$d\chi_t = a^X dt + b^X dW_t$$

with a^X, b^X to be constant.

TODO: Second-order of the corollary.

We can further generalize Lemma 1 to become:

Lemma 4. Consider a W-differentiable process χ_t defined above. Consider, also an independent Brownian motion W'_t. It holds that:

$$\mathbb{E}_0\left[e^{iuW_t}e^{ivW'_t}\chi_t\right] = e^{-\frac{u^2+v^2}{2}t}\left(\chi_0 + \int_0^t e^{\frac{u^2+v^2}{2}s}\mathbb{E}_0\left[e^{iuW_s}e^{ivW'_s}(d\chi_s + iud[\chi, W]_s)\right]\right)$$

where $[\chi, W]_t$ is the quadratic covariation process between χ_t and W_t.

We will not state Theorem 1, due to the complicated nature of the theorem. However, we will state Corollary 3, and attempt to understand the proof in the paper.

Corollary 3. If $X_t \in D_W^{(5)}$, the conditional characteristic function of Z_Δ with respect to time-0 information is expressed as follows:

$$\begin{aligned}\mathbb{E}_0\left[e^{iuZ_\Delta}\right] = e^{-\frac{u^2}{2}}\Bigg(1 &- iu^3\frac{\beta_0}{2\sigma_0}\sqrt{\Delta} - \frac{1}{2}u^2\frac{\alpha_0+\delta_0}{\sigma_0}\Delta \\ &- \frac{1}{8}\frac{\beta_0^2}{\sigma_0^2}u^2\left(2-4u^2+u^4\right)\Delta \\ &+ \frac{(\beta'_0)^2}{\sigma_0^2}\left(\frac{1}{6}u^4 - \frac{1}{4}u^2\right)\Delta \\ &+ \frac{\eta_0}{6\sigma_0}u^4\Delta\Bigg) + O(\Delta^{3/2})\tilde{\phi}(u),\end{aligned}$$